Grade 6 Math Circles

February 5th, 2024

Number Theory: Divisibility and Proofs - Problem Set

Note: Problems that are marked with * are considered challenge problems!

1. List all the positive whole numbers that are divisors of the following numbers;
   (a) 60
   (b) 180

> *Solution*:
> (a) The divisors of 60 are- 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60
> (b) The divisors of 180 are- 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180

2. List all the positive whole numbers that are divisors of 3, 9, 27, and 81. What do you notice?

> *Solution*: The divisors of 3 are 1 and 3, the divisors of 9 are- 1, 3, and 9, the divisors of 27 are- 1, 3, 9, and 27, lastly the divisors of 81 are- 1, 3, 9, 27, and 81. From this we can notice that each number contains the divisors of the previous number plus itself, that is the divisors of 9 are the divisors of 3 and 9, and the divisors of 27 are the divisors of 9 and 27. we can notice that $9 = 3 \times 3$, $27 = 3 \times 3 \times 3$, and $81 = 3 \times 3 \times 3 \times 3$. This repeated multiplication is what creates these chains of divisors, and these chains can be created by using any whole numbers!!

3. Use the definition of divisibility to show that all numbers divide 0.

> *Solution*: To show that all numbers divide 0, we first recall the definition of divisibility from the lesson, which says- $x \mid y$ if we can find a whole number z so that $x \times z = y$, so our goal is to find a whole number z so that $x \times z = 0$. But since the product of any number times 0 is 0- that is $x \times 0 = 0$ this shows that we were able to find a whole number z so that $x \times z = 0$. thus all numbers divide 0 as desired!

4. Use the Rule for Divisibility by 3 to determine if the following statements are true or false:

(a) $3 \mid 81$

(b) $3 \mid 111111111111111111111$

(c) $3 \mid 222222222$

(d) $3 \mid 1293746$

(e) $3 \mid 343293$

Solution:

(a) True: $8 + 1 = 9$ and we know that $3 \mid 9$ so $3 \mid 81$

(b) True: $1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1 = 21$ and $2 + 1 = 3$ since $3 \mid 3$ then $3 \mid 21$ and since $3 \mid 21$ then $3 \mid 111111111111111111111$.

(c) True: $2+2+2+2+2+2+2+2+2 = 18$ and $1+8 = 9$ since $3 \mid 9$ then $3 \mid 18$ and since $3 \mid 18$ then $3 \mid 222222222$.

(d) False: $1 + 2 + 9 + 3 + 7 + 4 + 6 = 32$ and $3 + 2 = 5$ since $3 \nmid 5$ then $3 \nmid 32$ and so $3 \nmid 1293746$.

(e) True: $3+4+3+2+9+3 = 24$ and $2+4 = 6$ since $3 \mid 6$ then $3 \mid 24$ and since $3 \mid 24$ then $3 \mid 343293$.

5. Use the Rule for Divisibility by 4 to determine if the following statements are true or false:

(a) $4 \mid 17$

(b) $4 \mid 222222222$

(c) $4 \mid 1293744$

(d) $4 \mid 18318716$

(e) $4 \mid 3432908$

Solution:

(a) False

(b) False: $4 \nmid 22$ so $4 \nmid 222222222$

(c) True: $4 \mid 44$ so $4 \mid 1293744$

(d) True: $4 \mid 16$ so $4 \mid 18318716$

(e) True: $4 \mid 8$ so $4 \mid 3432908$- notice in this example the last two digits make up the number 08 which is not a “real” two-digit number- so we let $08 = 8$.

6. Use the Rule for Divisibility by 5 to determine if the following statements are true or false:
   (a) $5 \mid 25$
   (b) $5 \mid 117365$
   (c) $5 \mid 1293744$

Solution:
(a) True: 25 ends in 5 so $5 \mid 25$
(b) True: 117365 ends in 5 so $5 \mid 117365$
(c) False: 1293744 does not end in 0 or 5 so $5 \nmid 1293744$

7. Use the Rule for Divisibility by 6 to determine if the following statements are true or false:
   (a) $6 \mid 24$
   (b) $6 \mid 1173657$
   (c) $6 \mid 1000100100$
   (d) $6 \mid 1293744$

Solution:
(a) True: 24 ends in 4 so it is even, and $2 + 4 = 6$ and since $3 \mid 6$ then $3 \mid 24$. So 24 is even and divisible by 3 so $6 \mid 24$
(b) False: 1173657 ends in 7 so it is not even thus $6 \nmid 1173657$
(c) True: 1000100100 ends in 0 so it is even, and $1+0+0+0+1+0+0+1+0+0 = 3$ since $3 \mid 3$ then $3 \mid 1000100100$, Thus since 1000100100 is even and $3 \mid 1000100100$ then $6 \mid 1000100100$
(d) False: 129374 ends in 4 so it is even, but $1 + 2 + 9 + 3 + 7 + 4 = 26$ and $2 + 6 = 8$ since $3 \nmid 8$ then $3 \nmid 26$ and thus $3 \nmid 129374$. Thus $6 \nmid 129374$.

8. Use the Rule for Divisibility by 7 to determine if the following statements are true or false:
   (a) $7 \mid 84$

(b) $7 \mid 365$

(c) $7 \mid 10000$

(d) $7 \mid 11111$

(e) $7 \mid 1293744$

Solution:

(a) True: the ones digit of 84 is 4 so $2 \times 4 = 8$ then $8 - 8 = 0$ and we know $7 \mid 0$, so $7 \mid 84$

(b) False: The ones digit is 5 so $2 \times 5 = 10$ then $36 - 10 = 26$ and we know $7 \nmid 26$ thus $7 \nmid 365$

(c) False: The ones digit is 0 so $2 \times 0 = 0$ then $100 - 0 = 100$. We repeat the rule of 7 once again- the The ones digit is 0 so $2 \times 0 = 0$ then $10 - 0 = 10$ from this we know $7 \nmid 10$ and so $7 \nmid 100$ which allows us to conclude that $7 \nmid 1000$.

(d) False: just like in part b we will be applying the rule of divisibility by 7 multiple times-

- In the first application the ones digit of 11111 is 1 so $2 \times 1 = 1$ then $1111 - 2 = 1109$.
- In the second application the ones digit of 1109 is 9 so $2 \times 9 = 18$ then $110 - 18 = 92$.
- In the third application the ones digit of 92 is 2 so $2 \times 2 = 4$ then $9 - 4 = 5$.

Since $7 \nmid 5$ then $7 \nmid 92$, since $7 \nmid 92$ then $7 \nmid 1109$ and finally since $7 \nmid 1109$ then $7 \nmid 11111$.

(e) False: just like in part c we will be applying the rule of divisibility by 7 multiple times-

- In the first application the ones digit of 1293744 is 4 so $2 \times 4 = 8$ then $129374 - 8 = 129366$.
- In the second application the ones digit of 129366 is 6 so $2 \times 6 = 12$ then $12936 - 12 = 12924$.
- In the third application the ones digit of 12924 is 4 so $2 \times 4 = 8$ then $1292 - 8 = 1284$.

- In the fourth application the ones digit of 1284 is 4 so $2 \times 4 = 8$ then $128 - 8 = 120$.
- In the fifth application the ones digit of 120 is 0 so $2 \times 0 = 0$ then $12 - 0 = 12$.

Since $7 \nmid 12$ then $7 \nmid 120$, similarly since $7 \nmid 120$ then $7 \nmid 1284$, once again since $7 \nmid 1284$ then $7 \nmid 12924$, and finally since $7 \nmid 12924$ then $7 \nmid 1293744$.

9. Use the Rule for Divisibility by 8 to determine if the following statements are true or false:
   (a) $8 \mid 16$
   (b) $8 \mid 365$
   (c) $8 \mid 10000$
   (d) $8 \mid 11111$
   (e) $8 \mid 1293744$

Solution:
(a) True: $8 \times 2 = 16$
(b) False: since 365 is odd then $8 \nmid 365$
(c) True: 0 makes up the last three digits of 10000 and since $8 \mid 0$ then $8 \mid 10000$
(d) False: since 11111 is odd then $8 \nmid 11111$
(e) True: 744 makes up the last three digits of 1293744 so all we have to do is check that $8 \mid 744$ unfortunately there is no short cut to figure out if this is true so we will have to use regular long division. The point of this is to see that at times math will require you to work through some rough calculations. Once you preform long division you get that $8 \times 93 = 744$ so $8 \mid 744$ and thus $8 \mid 1293744$.

10. Use the Rule for Divisibility by 9 to determine if the following statements are true or false:
   (a) $9 \mid 18$
   (b) $9 \mid 36582$
   (c) $9 \mid 10100100100101010100001000$
   (d) $9 \mid 11112$
   (e) $9 \mid 1293744$

Solution:

(a) True: $1 + 8 = 9$ and $9 \mid 9$ so $9 \mid 18$

(b) False: $3 + 6 + 5 + 8 + 2 = 24$ and $2 + 4 = 6$ since $9 \nmid 6$ then $9 \nmid 24$ and thus $9 \nmid 36582$

(c) True: $1+0+1+0+0+1+0+0+1+0+0+1+0+1+0+1+0+1+0+0+0+0+1+0+0+0 = 9$ since $9 \mid 9$ then $9 \mid 10100100100101010100001000$

(d) False: $1 + 1 + 1 + 1 + 2 = 6$ since $9 \nmid 6$ then $9 \nmid 11112$

(e) False: $1 + 2 + 9 + 3 + 7 + 4 + 4 = 30$ and $3 + 0 = 3$ since $9 \nmid 3$ then $9 \nmid 30$ and so $9 \nmid 1293744$

11. Come up with rules for division by 18 and 24.

Solution:

- First we look at 18; just like our rule for 6 we want to first check if we can express 18 as the product of 2 whole numbers which do not share a common factor. $18 = 1 \times 18 = 2 \times 9 = 3 \times 6$, from this list we know 1 and 18 share a common factor of 1 and we also know that 3 and 6 share a common factor of 3 since $3 \mid 6$. But we know that 2 and 9 do not share a common factor! Thus the rule for divisibility by 18 will combine the rules for divisibility by 2 and 9, giving us the following rule: " Let x be any whole number then $18 \mid x$ if and only if x is even and the sum of its digits is a multiple of 9."
- now we'll look at 24; just like our rule for 6 we want to first check if we can express 18 as the product of 2 whole numbers which do not share a common factor. $24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6$. From this list we know 1 and 24 share a common factor of 1, 2 and 12 share a common factor of 2, and 4 and 6 share a common factor of 2. But we know that 3 and 8 do not share a common factor! Thus the rule for divisibility by 24 will combine the rules for divisibility by 3 and 8, giving us the following rule: "Let x be any whole number then $24 \mid x$ if and only if the the sum of the digits of x is a multiple of 3 and the last three digits of x make up a multiple of 8.

12. Use the rules of divisibility to fully factor 2520.

Solution: For clarity we will present our solution in steps, by starting with the smallest possible factor and working our way up;

(a) first we check if $2 \mid 2520$, since 2520 ends in 0 then $2 \mid 2520$. Once we divide by 2 we get that $2520/2 = 1260$. This gives us a factor of 2.

(b) Once again we'll check if $2 \mid 1260$, since 1260 ends in 0 then $2 \mid 1260$. Once we divide by 2 we get that $1260/2 = 630$. This gives us a factor of 2.

(c) Once more we'll check if $2 \mid 630$, since 630 ends in 0 then $2 \mid 630$. Once we divide by 2 we get that $630/2 = 315$. This gives us a factor of 2.

(d) This time notice that 315 ends in 5 so $2 \nmid 315$ This does not give us a factor.

(e) Now we'll check if $3 \mid 315$, $3+1+5 = 9$, and since $3 \mid 9$ then $3 \mid 315$. Once we divide by 3 we get that $315/3 = 105$. This gives us a factor of 3.

(f) Starting over again we check if $2 \mid 105$, since 105 ends in 5 we know that $2 \nmid 105$. This does not give us a factor.

(g) Now we'll check if $3 \mid 105$, $1+0+5 = 6$, and since $3 \mid 6$ then $3 \mid 105$. Once we divide by 3 we get that $105/3 = 35$. This gives us a factor of 3.

(h) Starting over again we check if $2 \mid 35$, since 35 ends in 5 we know that $2 \nmid 35$. This does not give us a factor.

(i) Now we'll check if $3 \mid 35$, $3 + 5 = 8$, and since $3 \nmid 8$ then $3 \nmid 35$. This does not give us a factor.

(j) since $2 \nmid 35$ then $4 \nmid 35$

(k) Now since 35 ends in 5 we know $5 \mid 35$, Once we divide by 5 we get that $35/5 = 7$. This gives us a factor of 5, and a factor of 7.

Therefore using our divisibility rules we have shown that $2520 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7$

13. * Let x be a 4 digit number. Prove that **if** $9 \mid x$, **then** the digits of x add up to a multiple of 9.

Solution: (Recall to prove an *if-then* statement we start by assuming that the hypothesis is true and work towards proving the conclusion)

Assume that $9 \mid x$ and let x by a four digit number which we can write as $x = 1000a +$

$100b + 10c + d$ then $9 \mid 1000a + 100b + 10c + d$ since $9 \mid x$. Our goal is to show that $9 \mid (a + b + c + d)$!
We first notice that we can rewrite $1000a + 100b + 10c + d$ as $1000a + 100b + 10c + d = 999a + 99b + 9c + (a + b + c + d)$. Now we will look at $999a + 99b + 9c$ applying our rule for divisibility by 9 we see that $9 \mid 9$ and so $9 \mid 9c$, $9 + 9 = 18$ and since $9 \mid 18$ then $9 \mid 99$ and so $9 \mid 99b$, lastly since $9 + 9 + 9 = 27$ and $2 + 7 = 9$ then $9 \mid 27$ and thus $9 \mid 999$ which gives us that $9 \mid 999a$ (we know that these claims hold by Property 2 of divisibility). Now applying Property 1 of divisibility we have that $9 \mid 999a + 99b + 9c$. So far we've show that $9 \mid 999a + 99b + 9c + (a + b + c + d)$ and $9 \mid 999a + 99b + 9c$ and so we can again apply Property 1 of divisibility as follows:
Since $9 \mid 999a + 99b + 9c + (a + b + c + d)$ and $9 \mid 999a + 99b + 9c$ then $9 \mid 999a + 99b + 9c + (a + b + c + d) - 999a + 99b + 9c$ this gives us exactly that $9 \mid (a + b + c + d)$ as desired!

14. ** Let x be 2 digit whole number. Prove that **if** $7 \mid x$, **then** the difference between $2 \times$ the ones digit of x and the remaining part of x is divisible by 7.

Solution: We first begin by assuming that the hypothesis is true. Assume that $7 \mid x$ and let x be a two digit whole number which we can write as $10a + b$. Our goal is to show that the difference between $2 \times$ the ones digit of x and the remaining part of x is divisible by 7.

The ones digit of x is given by b so $2 \times$ the ones digit of x is $2 \times b$ and and the remaining part of x is given by a, thus we want to show that $7 \mid a - (2 \times b)$.

From our assumption we have that $7 \mid x$ and thus $7 \mid 10a + b$, first we want to get a $-2 \times b$ to appear on the right-hand side of our divisibility expression, so we will use Property 2 of divisibility to make this happen; since $7 \mid 10a + b$ then $7 \mid -2 \times (10a + b)$ which gives us that $7 \mid (-20 \times a) - (2 \times b)$.

Now, we want to make an a appear on the right-hand side of our divisibility expression so we will use Property 1 of divisibility to make this happen; we know that $7 \mid 21$ and so $7 \mid 21 \times a$ therefore we have that $7 \mid (-20 \times a) - (2 \times b)$ and $7 \mid 21 \times a$ so by Property 1 of divisibility we have that $7 \mid (21 \times a) - (20 \times a) - (2 \times b)$ this is exactly

$7 \mid a - (2 \times b)$ as desired.

15. *** Let x be a six digit number given by $x = abcdef$. Show that **if** x is divisible by 101, **then** $(ab - cd + ef)$ is divisible by 101.

Solution: Assume that $101 \mid x$ since x is a six digit number we can rewrite x as $x = 100000a + 10000b + 1000c + 100d + 10e + f$.

our goal is to show that $101 \mid (10a + b) - (10c + d) + (10e + f)$ we first notice that our expression already contains $10e + f$ so it remains to make $10a + b$ and $-10c - d$ appear on the right hand side of our divisibility expression.

Lets first look at $100000a + 10000b$ to get $10a + b$ we can rewrite $100000a + 10000b$ as $99990a + 9999b + 10a + b$. Similarly, we can look at $1000c + 100d$ and rewrite our expression as $1010c + 101d - 10c - d$.
We now have that $x = 100000a + 10000b + 1000c + 100d + 10e + f = 99990a + 9999b + 1010c + 101d + (10a + b) - (10c + d) + (10e + f)$.

Now we will notice that $99990a + 9999b = 9999(10a + b)$ and that $1010c + 101d = 101(10c + d)$, after doing long division we can see that $101 \times 99 = 9999$ and clearly $101 \times 1 = 101$ thus $101 \mid 9999$ and $101 \mid 101$. Then Properties 1 and 2 of divisibility tell us that $101 \mid 9999(10a+b)+101(10c+d)$ that is $101 \mid 99990a+9999b+1010c+101d$.

Therefore we have that $101 \mid 99990a + 9999b + 1010c + 101d$ and $101 \mid 99990a + 9999b + 1010c + 101d + (10a + b) - (10c + d) + (10e + f)$ thus by applying Property 1 of divisibility once more we have that $101 \mid 99990a + 9999b + 1010c + 101d + (10a + b) - (10c + d) + (10e + f) - (99990a + 9999b + 1010c + 101d)$ which gives us exactly that $101 \mid (10a + b) - (10c + d) + (10e + f)$ as desired!